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ARE DUAL VARIABLES PRICES?
IF NOT, HOW TO MAKE THEM MORE SO

by

George B. Dantzig

TECHNICAL REPORT SOL 78-6
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Abstract

Actual prices in an economy reflect a number of institutional arrangements -- salaries, savings, taxes, loans, interest, transfer payments, profits, rents, and investment credits. These can be quite different from prices generated by a L.P. (Linear Program). The price of an item in the L.P. is the change in the objective value if an additional unit of the item is made available to the system. An unfortunate consequence is that any capacity (or labor) not fully used gets a zero price. The purpose of this paper is to show how to make a simple perturbation to the linear program, after it has been solved, so that the new dual variables behave more like actual prices. To do this we will need three assumptions:

(a) the unused part of capacity is worth zero and can be deleted from the system;

(b) an infinitesimal ϵ part of the used capacity is malleable;

epsilon

(c) the value of capacity can be measured by deleting the malleable ϵ part and seeing what it is worth to put it back.

We shall show that it is possible to associate new prices with the optimal solution to the perturbed linear program without changing the original optimal primal solution. The new prices remain invariant as the malleable ϵ part of used capacity tends to zero.

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ARE DUAL VARIABLES PRICES?

IF NOT, HOW TO MAKE THEM MORE SO

I. The Method

The linear program can be taken in the form:

				Dual Variables
Find Max Z, $X \geq 0$:				
(1)	AX $-$	$b\theta$ ≥ 0		$-\pi$
(2)	$-IX$	$\geq -K$		$-\sigma$
		$1 \cdot \theta = Z(Max)$		

The objective is to maximize a vector output from the system, the bill-of-goods vector $b \cdot \theta$. K is the vector of capacities available to the system. It is understood that if certain variables have no upper bounds, these components are omitted from (2). The above formulation includes a shared capacity like labor; this is done by first summing up the total use of labor (say) and then placing a bound on this total in (2).

Suppose $X = X^0$, $\theta = \theta^0$ is optimal and we replace K by X^0. Reoptimizing we see that the presence of zero slacks means the optimal prices are no longer unique. For this reason a small perturbation can be used to make the prices unique again.

Accordingly, let $X = X^0$, $\theta = \theta^0$ be the optimal solution.
We now formulate a perturbed problem for some $\epsilon > 0$.

<div style="margin-left:2em;">

				Dual Variables

Find Max Z, $X \geq 0$, $Y \geq 0$:

</div>

(4) $\qquad AX - b\theta \geq 0 \qquad\qquad -\pi$

(5) $\qquad -IX \quad - IY \geq -X^0 \qquad -\sigma$

(6) $\qquad\qquad + \lambda Y \geq \epsilon \qquad\qquad -\rho$

$$\theta = Z(\text{Max})$$

We are in effect removing infinitesimal amount ϵ of the used part of the capacity from the system and letting the model decide how to allocate the removal.

The exchange rates for the different types of capacities i, when converted to some common unit, are assumed to be proportional to $(\lambda_1, \ldots, \lambda_n)$. For example, the λ_i could be proportional to the cost of building one unit of capacity in some historical base year. We refer to (6) as the <u>malleability constraint</u>.

The dual states, among other things, that the price on the i-th capacity:

(7) $\qquad\qquad \sigma_i \geq \rho \cdot \lambda_i \; .$

In applications, one can expect that <u>all</u> the used capacities are needed. One can arrange matters so that $\rho > 0$ and therefore $\sigma_i > 0$ for all i. (This will be shown after the following paragraph.)

The prices depend on ϵ but remain invariant for all $0 < \epsilon < \epsilon_1$ for some $\epsilon_1 > 0$. The proof is quite simple. The prices depend only on the choice of feasible basis. If the same basis is feasible for $\epsilon_1 > 0$ and $\epsilon_2 > 0$, $\epsilon_1 > \epsilon_2$, then it remains feasible for all ϵ: $\epsilon_1 \geq \epsilon \geq \epsilon_2$. Since there are only a finite number of different bases, it is clear that at least one basis will be repeated an infinite number of times as $\epsilon \to 0^+$. Hence, the basis choice (and hence the prices) will remain invariant for <u>all</u> ϵ in an interval $0 < \epsilon < \epsilon_1$ for some sufficiently small $\epsilon_1 > 0$.

One can construct examples, however, where $\rho = 0$. If so, we would replace the original problem by one that achieves $\theta = \theta^0$ but uses as little capacity as possible, for example:

Max Z, $X \geq 0$, $Y \geq 0$:

(8) $\qquad\qquad AX \qquad \geq b\theta^0$

(9) $\qquad\qquad -IX - IY \geq -K$

(10) $\qquad\qquad\qquad \lambda Y = Z(\text{Max})$

This will guarantee $\rho > 0$ if we use the new optimal solution, $X = X^0$, to initiate (4), (5), and (6).

Three Examples

We shall illustrate the approach on three examples. The first differs from the second in how much goods must be exported to receive the same quantity of imported oil. The third example shows how the Entitlement Policy of U.S. (which averages foreign and domestic oil

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prices) can dampen in a significant way the effects of rising oil import prices. The first two examples also illustrate how trivial changes in the amount of capacity available can have a dramatic effect on L.P. prices whereas the proposed method is not sensitive to such changes.

Example 1. In our little economy there are three industries, Energy (OIL), Manufacturing (MFG), Services (SER), whose capacities are 1.0, 1.07, 1.07 resp. There is a favorable balance-of-trade constraint that requires at least one unit of MFG to be exported for each unit of OIL imported. There is also a labor constraint. The final consumer (CONSM) receives some multiple θ of a fixed bill-of-goods vector. The objective (OBJ) is to maximize $Z = \theta$. See Table I.

At the optimum, both oil and labor capacities are tight; the levels of OIL, MFG, SER, Z are each 1.0 and imports of oil (IMP) = exports of manufactured goods (EXP) = .25.

The dual variables (π_1, π_2, π_3) and the bill of goods vector, $(\theta b_1, \theta b_2, \theta b_3) = (1.0, .5, .5)$, satisfy

$$\pi_1(\theta b_1) + \pi_2(\theta b_2) + \pi_3(\theta b_3) = Z \quad ,$$

the value of consumption. If we denote the capacity vector, $K = (K_5, \ldots, K_8) = (1.75, 1.00, 1.07, 1.07)$, then the value of capacity also equals the value of consumption (duality theorem):

$$\pi_5 K_5 + \pi_6 K_6 + \pi_7 K_7 + \pi_8 K_8 = Z$$

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where σ in our earlier notation is $(\pi_5, \pi_6, \pi_7, \pi_8)$. If we multiply the capacity constraints by $\pi_5, \ldots, \pi_8$ respectively and sum, we obtain

$$VA_{OIL} \cdot X_{OIL} + VA_{MFG} \cdot X_{MFG} + VA_{SER} \cdot X_{SER} = Z$$

where, for example, VA_{OIL} is (the value of labor + value of capacity) per unit of output of domestic oil production, usually referred to as value added. Note =Z not $\leq Z$ because complementary slackness conditions hold. Interpreting value added as salaries and profits (rents) paid to the final consumer by each industry -- we see the amount received equals the amount spent by the final consumer.

Let us assume for the moment that the prices on commodities $\pi_1 = .5$, $\pi_2 = .5$, $\pi_3 = .5$ and the foreign exchange ratio $\pi_4 = .5$, are quite plausible as is $\pi_5 = .5$, the price on labor. Note that MFG and SER capacities although almost fully utilized have zero value $\pi_7 = \pi_8 = 0.0$. Let us suppose the value-added values (.25, .25, .50) also are plausible. Let us see what happens when we slightly increase the amount of labor available, say from 1.75 to 1.76 and decrease MFG capacity from 1.07 to 1.00. These changes do not alter the primal solution but they do greatly alter the dual solution. Compare the π and π' columns and VA and VA' rows in Table 1. The dual prices no longer look reasonable. Services have no value, labor has no value, $VA'_{SER} = 0$. It is clear that the L.P. prices can be very sensitive to trivial changes in availabilities.

Table 1

(Example 1, Unadjusted Prices)

I-O TABLEAU OF AN ECONOMY

MFG Exports = OIL Imports

	OIL	MFG	SER	IMP	EXP	CONSM	RHS	π	π'[1]
OIL	1	-.25	0	1		-1.00	$\geq$ 0	.50	.67
MFG	-.25	1	0		-1	-.50	$\geq$ 0	.50	.67
SER	-.25	-.25	1			-.50	$\geq$ 0	.50	0
T/B	-	-		-1	+1		$\geq$ 0	.50	.67
LABOR	-.25	-.50	-1				$\geq$ -1.75	.50	-
CAP-OIL	-1.						$\geq$ -1.00	.125	.5
CAP-MFG		-1					$\geq$ -1.07	-	.5
CAP-SER			-1				$\geq$ -1.07	-	-
OBJ					+1		= Z(MAX)	1	1

[SOL-X] [1.0][1.0][1.0][.25][.25] [1.0]   SOL = PRIMAL SOLUTION

VA   .25   .25   .50   (See note below)

VA'   .50   .50   -

NOTE: VA_j is the "value added" of the j-th industry per unit output:
labor input × price of labor + capacity input × price on capacity.

_____

[1] The π' prices are obtained by changing RHS:LABOR from 1.75 to 1.76 and CAP-MFG from 1.07 to 1.00. VA' are the values of value added using prices π' instead of π.

Table 2 illustrates the adjustment of prices. Labor available is replaced by labor used (1.75); OIL, MFG, and SER capacities are replaced by (1.0, 1.0, 1.0), the amounts of these capacities used by the optimal solution. An ε amount of the used capacity is removed from the system. Malleability is assumed expressed by the equation

$$.57143 \, \Delta L + 1.6 \, \Delta 0 + 1.0 \, \Delta M + 1.0 \, \Delta S = \varepsilon$$

which converts ΔL change of labor, $\Delta 0$ change of oil capacity, etc. to some common unit -- the total amount of which is ε. In the model we have stated the above as LHS $\geq \varepsilon$ but it is clear that at Max Z, LHS $= \varepsilon$. In practice a small value of ε is assigned, for example, $\varepsilon = .001$. If there is some doubt that the basic solution will remain feasible for all $0 < \varepsilon < .001$ then try a lower value, say $\varepsilon / 2$, to see if the same basis is obtained. If no, one halves again recursively until yes. If yes, then one extrapolates to see if the basic solution remains feasible as $\varepsilon \to 0$. If not, the extrapolation or $\varepsilon / 2$ provides a lower ε to try again. This test process is finite by our earlier proof. Alternatively parametric programming may be applied.

Table 2

(Example 1, Adjusted Prices)

EXPORTS = IMPORTS

	OIL	MFG	SER	IMP	EXP	CONSM	ΔL	ΔO	ΔM	ΔS	RHS		ADJ
OIL	1	-.25	0	1		-1.0					≥	0	.52146
MFG	-.25	1	0		-1	-.5					≥	0	.52146
SER	-.25	-.25	1			-.5					≥	0	.43562
Trade Balance				-1	+1						≥	0	.52146
Labor	-.25	-.50	-1.0				-1				≥	-1.75	.30750
Capacity { OIL	-1							-1			≥	-1.0	.20500
MFG		-1							-1		≥	-1.0	.12812
SER			-1							-1	≥	-1.0	.12812
ε-MAL							.57143	1.6	1.0	1.0	≥	ε	.12812
OBJ						1.0					=	Z(Max)	
[SOL-X]	[1]	[1]	[1]	[.25]	[.25]	[1]							
ADJ.VA	.28188	.28188	.43562										

← Infinitesimal →
Malleability
Adjustment

$\epsilon > 0 \quad \epsilon \to 0^+$

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Example 2. This example is identical with that of Table 1 except the trade balance relation reads

$$-2\, X_{IMP} + 1 \cdot X_{EXP} > 0$$

i.e., at least twice the amount of MFG goods is required to be exported per barrel of OIL imported. The comparative optimum levels of production and consumption are:

	X.OIL	X.MFG	X.SER	X.IMP	X.EXP	X.CON
Example 1	1.0	1.0	1.0	.25	.25	1.0
Example 2,3	1.0	1.059	.971	.176	.352	.912

The effect of doubling the amount of MFG required to be exported per unit of oil imported, is to reduce the gross national consumption Z from 1.0 to 0.912.

Again it is seen as summarized in Table 3 that prices in the L.P. are extremely sensitive to trivial changes in the availabilities of labor and capacities. Compare π and π' columns; also VA and VA'.

The prices are adjusted in Table 4 using the above levels of production in place of the given capacity and ε-malleability of used capacity. These prices are compared in Table 3 with unadjusted prices. A comparison of some of the adjusted prices is given below:

	Example 1	Example 2
Price of Oil	.52	.73
Value-added Oil	.28	.60
Foreign Exchange	.52	.37
Price of Labor	.31	.06

Table 3

COMPARATIVE PRICES FOR THE THREE EXAMPLES

	EXAMPLE 1 EXPORTS = IMPORTS			EXAMPLE 2 EXPORTS = 2 × IMPORTS			EXAMPLE 3 ENTITLEMENTS
	π	π'	ADJπ	π	π'	ADJπ	ADJπ
Goods							
OIL	.50	.67	.52	.71	.80	.73	.60
MFG	.50	.67	.52	.35	.40	.37	.48
SER	.50	–	.44	.24	–	.17	.31
Foreign Exchange	.50	.67	.52	.35	.40	.37	.48
Capacity							
LABOR	.50	–	.31	.23	–	.06	.11
OIL	.125	.50	.21	.50	.70	.58	.32
MFG	–	.50	.13	–	.20	.11	.20
SER	–	–	.13	–	–	.11	.20
Value Added							
OIL	.25	.50	.28	.55	.70	.60	.34
MFG	.25	.50	.28	.12	.20	.14	.25
SER	.50	–	.43	.24	–	.17	.31

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Table 4

(Example 2)

EXPORTS = 2 × IMPORTS

Adjusted Prices

	OIL	MFG	SER	IMP	EXP	CONSX	ΔL	ΔO	ΔM	ΔS		RHS	ADJ.π
OIL	1	-.25	0	1		-1.0					≥	0	.7315
MFG	-.25	1	0		-1	-.5					≥	0	.3658
SER	-.25	-.25	1			-.5					≥	0	.1712
Trade Balance	-	-	-	-2	+1						≥	0	.3658
Labor	-.25	-.50	-1.0				-1				≥	-1.75	.0623
Capacity { OIL	-1							-1			≥	-1.00	.5817
Capacity { MFG		-1							-1		≥	-1.059	.1090
Capacity { SER			-1							-1	≥	-.971	.1090
ε-MAL							.57143	1.60	1.0	1.0	≥	ε	.1090
OBJ						1.0					=	Z(MAX) 1	
[SOL-X]	[1.000]	[1.059]	[.971]	[.176]	[.352]	[.912]							
ADJ.VA	.5973	.1401	.1712										

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Example 3. This example is identical with Example 2 except an "Entitle-ment" policy is now in effect which requires that the oil industry as a whole, domestic plus imports but not separately, to balance the books. Some of the profits of domestic oil production are transferred to cover losses incurred by those importing oil. The way this is done in the model is to combine the oil production column and the oil import column into a single column using as weights the optimum levels of oil produc-tion (1.0) and imports (.17647) attained by the solution of Example 2:

	OIL		IMP		OIL + IMP
OIL	1		1		1.17647
MFG	-.25		0		-.25
SER	-.25	(1.0) +	0	(.17647) =	-.25
T/B	0		-2		-.35294
LABOR	-.25		0		-.25
CAP.OIL	-1		0		-1.0

The prices generated this way are no longer unique because (a) the prices generated in Example 2 are still optimal, (b) there is one degree of freedom because one less column with positive primal variables is required to price to zero. This one degree of freedom was used to make the new prices look as much as possible like those of Example 1, i.e., as much as they were before the relative price hike of imported oil took place. The model for adjusting prices is shown in Table 5. The adjusted prices are compared with other prices in Table 3. It is seen that an entitlement policy can significantly dampen price changes due to a rise in oil import prices relative to export prices.

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Table 5

(Example 3, Entitlements)

EXPORTS = 2 × IMPORTS

Adjusted Prices

	OIL IMPORT + DOMESTIC	MFG	SER	EXP	CONSM	ΔL	ΔO	ΔM	ΔS		RHS	ADJ π
OIL	1.17647	− .25	0		−1.0					≥	0	.60440
MFG	− .25	1	0	−1	− .5					≥	0	.48170
SER	− .25	− .25	1.0		− .5					≥	0	.30949
TB	− .35294	−	−	+1						≥	0	.48170
Labor	− .25	− .50	−1.0			−1.0				≥	−1.75	.11254
CAP-O	−1.0						−1.0			≥	−1.00	.31512
CAP-M		−1.0						−1.0		≥	−1.05882	.19695
CAP-S			−1.0						−1.0	≥	− .97059	.19695
ε-MAL						.57143	1.6	1	1	≥	ε	.19695
OBJ					1.0					=	Z(Max)	1
[SOL.X]	[1.0]	[1.05882]	[.97059]	[.35294]	[.91176]							
V.A.	.34326	.25322	.30949									

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